On The Decidability Of MELL: Reachability In Petri Nets With Split/Join Transitions☆

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Abstract

We define Petri Nets with Split/Join transitions, a new model that extends Petri nets. We prove that reachability in this model without join transitions is equivalent to the decidability of MELL. We define a suitable notion of covering graph for the model, and prove its finiteness and effective constructibility.

Introduction

Petri nets, and equivalent models of computation such as Vector Addition Systems (VAS) and Vector Addition Systems with States (VASS) are a natural parallel and resource sensitive model of computation, for which reachability, among other key properties, has been proven decidable [Kos82, May84, May81, Lam92, Reu88]. After the introduction of Linear Logic by Girard [Gir87], several authors focused on the syntactical and semantical relations between Linear Logic and Petri Nets [MOM89, Bro90, AFG90, Far99]. Results more relevant to our current considerations has been obtained by Kanovich, who established several equivalence results between decision problems for various kinds of Petri Nets and different fragments of Linear Logic [Kan92, Kan94, Kan96, Kan95]. In particular, he proved in [Kan95] the equivalence between the !-Horn fragment of Mutiplicative Exponential Linear Logic (MELL) and the reachability problem for Petri Nets, which is known to be decidable. His result, however, does not yield the decidability of MELL, a problem still open today.
A further step connecting MELL decidability and reachability properties for Petri Nets has been achieved by de Groote, Guillaume and Salvati [dGGS04], who established the equivalence between MELL and the reachability problem for Vector Addition Tree Automata (VATA), a generalization of Vector Addition Systems. Independently, Verma and Goubault-Larrecq introduced a branching extension of VASS, called BVASS, for which they proposed a notion of Karp and Miller Tree that allows to establish the decidability of properties such as finiteness, boundedness and emptiness of their model [VGL05]. Yet, they do not obtain the decidability or undecidability of the reachability problem. It turns out that the BVASS model is actually equivalent to the VATA model of de Groote, Guillaume and Salvati: therefore, the work of Verma and Goubault-Larrecq is indeed an interesting step towards the decidability of MELL. Further work on the complexity of decision problems for the BVASS model has been done by Demri [DJLL09].
A careful reading of the reachability algorithm of Kobayashi and Mayr [Kos82, May84, May81] for Petri Nets or for VASS reachability reveals a feature of the model that is central in the construction of the algorithm: Petri Nets, as well as its equivalent models, VAS and VASS, are symmetric, in the sense that the model is stable by an inversion of the arrows. This feature is not present in the VATA or in the BVASS model, hence it seems unlikely that one can adapt the algorithm of Kobayashi and Mayr to the setting of VATA or BVASS.

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Our contribution to this line of research consists in the symmetrization of the BVASS model, that we call Petri Nets with Split/Join transitions (PN-SJ) or, equivalently, Vector Addition Systems with States and Split/Join transitions (VASS-SJ). We provide a new proof of the result of de Groote, Guillaume and Salvati [dGGS04] in this setting, namely that MELL decidability is equivalent to the reachability problem for PN-SJ with no join transition. Our proof is in some way more elegant than that of de Groote, Guillaume and Salvati, in the sense that it clearly relates the use of tokens in PN-SJ with occurrences of formulae in MELL sequent calculus, and the semantics of split transitions in PN-SJ with the role played by the $\otimes$ rule in MELL sequent calculus. We also introduce a suitable extension of the classical notion of Karp and Miller Tree for VASS to this new model of VASS-SJ, that we call Karp and Miller Graph. In this setting, a Karp and Miller Graph is no more a tree, by a directed, acyclic graph. We prove its finiteness and effective constructibility, and prove its use for coverability results.
The paper is organized as follows: In Section 1, we define the model of Petri Nets with Split/Join transitions (PN-SJ) and its execution semantics. We give in Theorem 1.9 the equivalence between MELL decidability and the reachability problem for PN-SJ with no join transition. In Section 2, we define the model of Vector Addition System with States and Split/Join transitions (VASS-SJ), give its execution semantics, and prove its equivalence with PN-SJ. In Section 3, we define our notion of Karp and Miller Graph, prove its finiteness and effective constructibility in Theorem 3.5, and prove its use for coverability results in Theorem 3.6.

1. Petri Nets with Split/Join Transitions

1.1. Definition

Definition 1.1. Petri Nets with Split/Join Transitions.

A Petri Net with Split/Join transitions (PN-SJ) is a 4-tuple $N = (P, T, W^-, W^+)$, where

- P is a finite set of places
- T is a finite set of transitions
- $W^- : P \times T \to \mathbb{N} \cup \mathbb{N}^2$ is a pre-incidence function
- $W^+ : P \times T \to \mathbb{N} \cup \mathbb{N}^2$ is a post-incidence function,

such that:

1. $\forall t \in T, \forall p, p' \in P :$ arity$(W^-(p,t)) =$ arity$(W^-(p',t))$. This defines the pre-arity of t,
2. $\forall t \in T, \forall p, p' \in P :$ arity$(W^+(p,t)) =$ arity$(W^+(p',t))$. This defines the post-arity of t,
3. no transition t has both pre and post-arities equal to 2.

A transition t with pre-arity 2 is called a *join* transition, a transition t with post-arity 2 is called a *split* transition. A transition with pre and post -arity 1 is called a *regular* transition.
A PN-SJ with only regular and split transitions is called a *Petri Nets with Split transitions* (PN-S), and a PN-SJ with only regular and join transitions is called a *Petri Nets with Join transitions* (PN-J).
A *single marking* of N is a mapping $M : P \to \mathbb{N}$. We say the single marking assigns to each place a number of *tokens*. A *marking* of N is a finite multiset $\mathcal{M}$ of single markings.
For any $t \in T$, we will write $W^-(t)$ (respectively $W_1^-(t)$, $W_2^-(t)$) for the single marking $p \to W^-(p,t)$ (resp. $p \to W_1^-(p,t)$, $p \to W_2^-(p,t)$), and $W^+(t)$ (resp. $W_1^+(t)$, $W_2^+(t)$) for the single marking $p \to W^+(p,t)$ (resp. $p \to W_1^+(p,t)$, $p \to W_2^+(p,t)$).
A *marked* PN-SJ is a 5-tuple $((P, T, W^-, W^+, \mathcal{M})$, where

- $N = (P, T, W^-, W^+)$ is a PN-SJ,
- $\mathcal{M}$ is a marking of N.

1.2. Execution Semantics of a PN-SJ

Definition 1.2. Enabling of a transition.

Let $(N, \mathcal{M})$ be a marked PN-SJ. Let t be a transition of N. t is *enabled* in $(N, \mathcal{M})$ if and only if

- t is a regular transition and there exists $M \in \mathcal{M}$ such that, for all $p \in P$, $M(p) \geq W^-(p,t)$. In this case we say that t is enabled in M, or
- t is a split transition and there exists $M \in \mathcal{M}$ such that, for all $p \in P$, $M(p) \geq W^-(p,t)$. In this case we say that t is enabled in M, or
- t is a join transition and there exist $M_1, M_2 \in \mathcal{M}$ such that, for all $p \in P$, $M_1(p) \geq W^-(p,t)_1$ and $M_2(p) \geq W^-(p,t)_2$. In this case we say that t is enabled in M_1 and M_2.

Definition 1.3. Firing of a transition.

Let $(N, \mathcal{M})$ be a marked PN-SJ, and t be a transition of N enabled in $(N, \mathcal{M})$. The *firing* of t in $(N, \mathcal{M})$ is the relation

$$(N, \mathcal{M}) \rightarrow_t (N, \mathcal{M}'),$$

where

- if t is a regular transition enabled for one $M \in \mathcal{M}$, $M' = p \rightarrow M(p) - W^-(p,t) + W^+(p,t)$, and $\mathcal{M}' = \mathcal{M} \setminus \{M\} \uplus \{M'\}$.
- if t is a split transition enabled for one $M \in \mathcal{M}$, let M_1 and M_2 be two single markings of N such that , for all $p \in P$, $M_1(p) - W^+(p,t)_1 \geq 0$, $M_2(p) - W^+(p,t)_2 \geq 0$, and $M_1(p) + M_2(p) = M(p) - W^-(p,t) + W^+(p,t)_1 + W^+(p,t)_2$. Then, $\mathcal{M}' = \mathcal{M} \setminus \{M\} \uplus \{M_1\} \uplus \{M_2\}$.
- if t is a join transition enabled for M_1 and M_2, let $M' = p \rightarrow M_1(p) - W_1^-(p,t) + M_2(p) - W_2^-(p,t) + W^+(p,t)$ and $\mathcal{M}' = \mathcal{M} \setminus \{M_1\} \setminus \{M_2\} \uplus \{M'\}$.

In words,

- firing a regular transition t in a single marking M consumes $W^-(p,t)$ tokens from each of its input places p, and produces $W^+(p,t)$ tokens in each of its output places p,
- firing a split transition t in a single marking M consumes $W^-(p,t)$ tokens from each of its input places p, splits the marking M in two new markings M_1 and M_2, and produces $W^+(p,t)_1$ tokens in each of its output places p in M_1 and $W^+(p,t)_2$ tokens in each of its output places p in M_2, and
- firing a join transition t in a couple (M_1, M_2) of single markings consumes $W^-(p,t)_1$ tokens in each of its input places p in M_1 and $W^-(p,t)_2$ tokens in each of its input places p in M_2, sums the two markings into a new marking M', and produces $W^+(p,t)$ tokens in each of its output places p.

Remark 1.4. The execution semantics of a PN-SJ with only regular transitions is exactly that of a classical petri net.

Definition 1.5. Promenade.

Let N be a PN-SJ. A promenade on N is a labelled acyclic directed finite graph with in and out-degree at most two such that:

- each vertex is labelled with a single marking on N,
- each edge is labelled with a transition of N,

- for any vertex, the (possibly two) ingoing edges have the same label, as well as the (possibly two) outgoing edges,
- for any vertex v labelled with M, any outgoing edge e labelled with t, t is enabled in M,
- for any vertex v labelled with M, with indegree two, with parent nodes v_1 labelled with M_1 and v_2 labelled with M_2, the ingoing edges are labelled with a join transition t, and we have:

$$(N, \{M_1\} \uplus \{M_2\}) \rightarrow_t (N, \{M\}),$$

- for any vertex v' labelled with M', with indegree one, with parent node v labelled with M with outdegree one, the ingoing edge is labelled with a regular transition t, and we have:

$$(N, \{M\}) \rightarrow_t (N, \{M'\}),$$

- for any vertex v labelled with M, with outdegree two, with child nodes v_1 labelled with M_1 and v_2 labelled with M_2, the outgoing edges are labelled with a split transition t, and we have:

$$(N, \{M\}) \rightarrow_t (N, \{M_1\} \uplus \{M_2\}).$$

Definition 1.6. Reachability Problem.

Let N be a PN-SJ, $\mathcal{M}_0$, $\mathcal{M}_1$ be two sets of single markings on N. The reachability problem for N, $\mathcal{M}_0$, $\mathcal{M}_1$ is the following:
Does there exists a promenade P on N such that:

1. the set of labels of vertices of indegree 0 is $\mathcal{M}_0$, and
2. the set of labels of vertices of outdegree 0 is $\mathcal{M}_1$?

The reachability problem for N, $\mathcal{M}_0$, $\mathcal{M}_1$ can easily be reformulated as follows:
Does there exist two markings $\mathcal{M}'_0$, $\mathcal{M}'_1$ such that:

1. the underlying set of $\mathcal{M}'_0$ is $\mathcal{M}_0$,
2. the underlying set of $\mathcal{M}'_1$ is $\mathcal{M}_1$, and
3. there exists a finite sequence of transitions $t_0, \cdots, t_k$ of N such that

$$(N, \mathcal{M}'_0) \rightarrow_{t_0} \cdots \rightarrow_{t_k} (N, \mathcal{M}'_1).$$

1.3. Correspondence between MELL *and the PN-S Reachability Problem*
Roman capitals A, B stand for MELL formulas, which are given by the following grammar, where $\otimes$ and $⅋$ are duals, ! and ? are duals, and the neutral elements 1 and $\perp$ are duals for the negation $^\perp$ accordingly to De Morgan laws:

$$\text{MELL:} \qquad F ::= A \mid A^\perp \mid F \otimes F \mid F ⅋ F \mid\, !F \mid\, ?F \mid 1 \mid \perp$$

Greek capitals Γ, Δ stand for sequents, which are multiset of formulas, so that exchange is implicit. The MELL (cut free) sequent calculus is given by the following rules:

$$\frac{}{\vdash A, A^\perp}\,(ax) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A ⅋ B}\,⅋ \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\,\otimes$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, ?A}\,?W \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\,?C \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\,?D \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\,!P$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}\,\perp \qquad \frac{}{\vdash 1}\,1$$

The MELL decision problem is to decide, given a MELL sequent Γ, wether there exists a sequent calculus proof with conclusion $\vdash \Gamma$. We will simply write $\vdash \Gamma$ for Γ satisfies the MELL problem.

Lemma 1.7. *For any of the* $\parr, ?W, ?C, ?D, \perp$ MELL *sequent calculus rules* (R) *applied on a formula* f *with conclusion* Γ_0 *and premise* Γ_1*, any* MELL *formula* A*,* (R) *can be applied on* f *with conclusion* $\Gamma_0 \cup A$ *and premise* $\Gamma_1 \cup A$*.*
For any $\otimes$ *sequent calculus rules* (R) *applied on a formula* f *with conclusion* Γ_0 *and premises* Γ_1 *and* Γ_2*, any* MELL *formula* A*,* (R) *can be applied on* f *with conclusion* $\Gamma_0 \cup A$ *and premises* $\Gamma_1 \cup A$ *and* Γ_2*, or on* f *with conclusion* $\Gamma_0 \cup A$ *and premises* Γ_1 *and* $\Gamma_2 \cup A$*.*

□

Lemma 1.8. *Let* Γ *be a* MELL *sequent with* $\vdash \Gamma$*, let* $F(\Gamma)$ *be the set of sub-formulas of the formulas of* Γ*, and* $\mathcal{P}(F(\Gamma))$ *be the set of subsets of* $F(\Gamma)$*.*
There exists a MELL *proof* Π *with conclusion* $\vdash \Gamma$ *such that:*

1. *for any !P sequent calculus rule* (R) *applied with conclusion* ζ *in* Π*,* ζ *equals its underlying set* Δ*,*
2. *for any !P sequent calculus rule* (R_1) *applied with conclusion* ζ *in* Π*, any !P sequent calculus rule* (R_2) *applied with conclusion* ζ' *in* Π *below* (R_1) *,* $\zeta \neq \zeta'$*, and*
3. *in any path of* Π*, the number of !P rules is bounded by* $|\mathcal{P}(F(\Gamma))|$*.*

Proof. Assume $\vdash \Gamma$, and let Π be a MELL proof Π with conclusion $\vdash \Gamma$.

1. Let (R) be a !P sequent calculus rule applied with conclusion ζ in Π. Then, it suffices to apply as many $?W$ rules as necessary under (R), and as many $?C$ rules as necessary above (R).
2. Let (R_1) and (R_2) be two !P rules applied in Π with the same conclusion ζ, (R_2) below (R_1). Then, one can remove in Π the tree above (R_2) and replace it by the tree above (R_1): the tree Π' obtained is still a MELL proof with conclusion $\vdash \Gamma$. Doing so for all such pairs of !P rules (R_1) and (R_2) yields the result.
3. follows directly from 1) and 2) above.

□

Theorem 1.9. *The PN-S Reachability Problems and* MELL *reduce one to the other via many-one reductions.*

Proof. Let us first prove that PN-S Reachability $\preceq_1$ MELL.
Let $N = (P, T, W^-, W^+)$ be a PN-S, $\mathcal{M}_0$, $\mathcal{M}_1$ be two sets of single markings on N. Since N is a PN-S, i.e. has no join transition, we can without loss of generality assume that $\mathcal{M}_0$ contains only one single marking M_0, and assume $\mathcal{M}_1 = \{M_1, \dots M_k\}$. Let us now define the sequent Γ as follows:

1. let $P = \{p_1 \cdots p_n\}$. To each place $p_i \in P$, we associate a linear variable p_i,
2. to a single marking M on N, we associate the formula $F(M) = \left(\bigotimes_1^{M(p_1)} p_1\right) \otimes \cdots \otimes \left(\bigotimes_1^{M(p_n)} p_n\right)$, where $\bigotimes_1^k a$ denotes $a \otimes \cdots \otimes a$ k times if $k \geq 1$, and 1 if $k = 0$,
3. to a regular transition r_i of N, we associate the following formula $R_i = (F(W^-(r_i))) \otimes (F(W^+(r_i)))^\perp$,
4. to a split transition s_j of N, we associate the following formula
$S_j = (F(W^-(s_j))) \otimes \left(\left(F(W_1^+(s_j))\right)^\perp \otimes \left(F(W_2^+(s_j))\right)^\perp\right)$, and finally
5. $\Gamma = \{?R_i\} \cup \{?S_j\} \cup F(M_0)^\perp \cup F(M_1) \cup \cdots \cup F(M_k) \cup ?F(M_1) \cup \cdots \cup ?F(M_k)$.

Then, $\vdash \Gamma$ if and only if N, $\mathcal{M}_0$, $\mathcal{M}_1$ satisfy the PN-S reachability problem.

Let us now prove that MELL $\preceq_1$ PN-S Reachability.
Let $\Gamma = F_1 \cup \cdots \cup F_k$ be a MELL sequent. Let $F(\Gamma)$ be the set of sub-formulas of the formulas of Γ, and $\mathcal{P}(F(\Gamma))$ be the set of subsets of $F(\Gamma)$.
Let ξ be a sequent with underlying set $\Xi \in \mathcal{P}(F(\Gamma))$, such that a !P rule can be applied on a formula $!A$ of ξ. Then, by Lemma 1.8, $\vdash \xi$ if and only if $\vdash \Xi$. Let $n(\Gamma) \leq |\mathcal{P}(F(\Gamma))|$ be the bound of Lemma 1.8, 3), and assume the possible !P rule applications in a proof of $\vdash \Gamma$ are numbered from 1 to $n(\Gamma)$, and their conclusion sequents $\Xi_1, \cdots \Xi_{n(\Gamma)}$. Let also $\mathcal{P} = \mathcal{P}(\{1 \cdots n(\Gamma)\})$. A set $\pi \in \mathcal{P}$ will be denoted as an *index* set (of sequents conclusions of !P rule applications).
We define the PN-S $N(\Gamma) = (P, T, W^-, W^+)$ as follows:

1. to every $\Delta \in \mathcal{P}(F(\Gamma))$, every $\pi \in \mathcal{P}$, we associate a place $p_{\Delta,\pi}$ in P. To every $f \in \Delta$, we associate a place $p^f_{\Delta,\pi}$ in P. The sets of places $\{p_{\Delta,\pi}, p^{f_i}_{\Delta,\pi};\ f_i \in \Delta\}$ encodes the underlying set of formulas of any occurrence of a sequent ζ that might occur in a proof with conclusion $\vdash \Gamma$, where π represents the index set of the conclusion sequents of all !P rules already present in the proof under the given occurrence of ζ. The multiplicity of a formula in such a sequent will be encoded by the number of tokens that will be present in the corresponding place. For any sequent ζ with underlying set Δ, any $\pi \in \mathcal{P}$, denote by $M(\zeta, \pi)$ the single marking encoding ζ, where:
   - $M(\zeta, \pi)(p_{\Delta,\pi}) = 1$,
   - $M(\zeta, \pi)(p^{f_i}_{\Delta,\pi})$ is the multiplicity of f_i in ζ, and
   - $M(\zeta, \pi)(p) = 0$ for any other place p.
2. Let $\Delta \in \mathcal{P}(F(\Gamma))$, ζ a MELL sequent with underlying set Δ, $\pi \in \mathcal{P}$, $f_i \in \zeta$ such that a sequent calculus rule (R_i) can be applied on f_i with conclusion $\vdash \zeta$.
   - Assume $(R_i) = (ax)$. To $(R_i), \Delta, \pi$, we associate a place $q_{(R_i,\Delta,\pi)}$, and
     (a) a regular transition which takes one token in $p_{\Delta,\pi}$, one token in $p^{f_i}_{\Delta,\pi}$, one token in $p^{f_i^\perp}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\pi)}$, and leaves all other places untouched (lock), and
     (b) a regular transition which takes one token in $q_{(R_i,\Delta,\pi)}$, and leaves all other places untouched (unlock).
   - Assume $(R_i) = 1$. To $(R_i), \Delta, \pi$, we associate a place $q_{(R_i,\Delta,\pi)}$, and
     (a) a regular transition which takes one token in $p_{\Delta,\pi}$, one token in $p^1_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\pi)}$, and leaves all other places untouched (lock), and
     (b) a regular transition which takes one token in $q_{(R_i,\Delta,\pi)}$, and leaves all other places untouched (unlock).
   - Assume $(R_i) = \parr$, $f_i = A \parr B$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. To $(R_i), \Delta, \Delta_1, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
     (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{A \parr B}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^A_{\Delta_1,\pi}$, one token in $p^B_{\Delta_1,\pi}$, and leaves all other places untouched (lock),
     (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched. This transition is called a *transfer*, and
     (c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock).
   - Assume $(R_i) = ?W$, $f_i = ?A$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. To $(R_i), \Delta, \Delta_1, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
     (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{?A}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, and leaves all other places untouched (lock),
     (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched (transfer), and
     (c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock).
   - Assume $(R_i) = ?C$, $f_i = ?A$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. To $(R_i), \Delta, \Delta_1, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
     (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{?A}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, two tokens in $p^{?A}_{\Delta_1,\pi}$, and leaves all other places untouched (lock),
     (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched (transfer), and

(c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock).

- Assume $(R_i) = ?D$, $f_i = ?A$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. To $(R_i), \Delta, \Delta_1, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
  (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{?A}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{A}_{\Delta_1,\pi}$, and leaves all other places untouched (lock),
  (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched (transfer), and
  (c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock).
- Assume $(R_i) = \perp$, $f_i = \perp$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. To $(R_i), \Delta, \Delta_1, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
  (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{\perp}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, and leaves all other places untouched (lock),
  (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched (transfer), and
  (c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock).
- Assume $(R_i) = \otimes$, $f_i = A \otimes B$, and let ζ_1, ζ_2 be the two premises of (R_i) with underlying sets Δ_1 and Δ_2. To $(R_i), \Delta, \Delta_1, \Delta_2, \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$ and a place $q_{(R_i,\Delta,\Delta_2,\pi)}$, and
  (a) a split transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{A\otimes B}_{\Delta,\pi}$, places on its first projection one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{A}_{\Delta_1,\pi}$, and on its second projection one token in $q_{(R_i,\Delta,\Delta_2,\pi)}$, one token in $p^{B}_{\Delta_2,\pi}$, and leaves all other places untouched (lock),
  (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{f_j}_{\Delta_1,\pi}$, and leaves all other places untouched (transfer),
  (c) for all $f_j \in \Delta \cap \Delta_2$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_2,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_2,\pi)}$, one token in $p^{f_j}_{\Delta_2,\pi}$, and leaves all other places untouched (transfer),
  (d) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{\Delta_1,\pi}$, and leaves all other places untouched (unlock), and
  (e) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_2,\pi)}$, places one token in $p_{\Delta_2,\pi}$, and leaves all other places untouched (unlock).
- Assume $(R_i) = !P$, $f_i = !A$, and let ζ_1 be the premise of (R_i) with underlying set Δ_1. Then, $\Delta = \Xi_j$ for some j. To $(R_i), \Delta, \Delta_1, \pi$ with $j \notin \pi$, we associate a place $q_{(R_i,\Delta,\Delta_1,\pi)}$, and
  (a) a regular transition which takes one token from $p_{\Delta,\pi}$, one token from $p^{!A}_{\Delta,\pi}$, one token from $p^{f_i}_{\Delta,\pi}$ for each of the $f_i \in \Delta \cap \Delta_1$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token in $p^{A}_{\Delta_1,\pi\uplus\{j\}}$, one token in $p^{f_i}_{\Delta,\pi\uplus\{j\}}$ for each of the $f_i \in \Delta \cap \Delta_1$, and leaves all other places untouched (lock),
  (b) for all $f_j \in \Delta \cap \Delta_1$, a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, one token from $p^{f_j}_{\Delta,\pi}$, places one token in $q_{(R_i,\Delta,\Delta_1,\pi)}$, and leaves all other places untouched (transfer), and
  (c) a regular transition which takes one token from $q_{(R_i,\Delta,\Delta_1,\pi)}$, places one token in $p_{(\Delta_1,\pi\uplus\{j\})}$, and leaves all other places untouched (unlock).

  To $(R_i), \Delta, \Delta_1, \pi$ with $j \in \pi$, we associate a place $q_{(sink)}$, and
  (a) a regular transition which takes one token from $p_{\Delta,\pi}$, places one token in $q_{(sink)}$,, and leaves all other places untouched (sink).

3. For each of the cases $(R_i) = (ax), 1, \parr, ?W, ?C, ?D, \perp, \otimes, !P$ above, the places and transitions associated only depend on f_i, Δ, π and on the underlying sets of the (possibly two) premises of the rule, Δ_1 and Δ_2. The construction above is repeated for any possible combination of possible sets Δ, Δ_1, Δ_2, π and formula f_i, where the number of such possible combinations is exponentially bounded.

Let $\emptyset$ be the empty single marking. Let us show that $\vdash \Gamma$ if and only if $N(\Gamma)$, $\{M(\Gamma, \emptyset)\}$, $\{\emptyset\}$ satisfies the PN-S reachability problem.
Firstly, assume $\vdash \Gamma$. Then, there exists a MELL sequent calculus proof Π with conclusion $\vdash \Gamma$. Assume Π satisfies the conditions of Lemma 1.8. Then,

1. for any rule application (R_i) in Π, with conclusion Γ_0 and zero premise, if π denotes the index set of conclusion sequents of the !P rules occurring in Π below (R_i), $N(\Gamma)$, $\{M(\Gamma_0, \pi)\}$, $\{\emptyset\}$ satisfies the PN-S reachability problem by the construction above.
2. For any rule application (R_i) other than !P in Π, with conclusion Γ_0 and one premise Γ_1, if π denotes the index set of conclusion sequents of the !P rules occurring in Π below (R_i), $N(\Gamma)$, $\{M(\Gamma_0, \pi)\}$, $\{M(\Gamma_1, \pi)\}$ satisfies the PN-S reachability problem by the construction above.
3. For any rule application (R_i) in Π, with conclusion Γ_0 and two premises Γ_1, Γ_2, if π denotes the index set of conclusion sequents of the !P rules occurring in Π below (R_i), $N(\Gamma)$, $\{M(\Gamma_0, \pi)\}$, $\{M(\Gamma_1, \pi), M(\Gamma_2, \pi)\}$ satisfies the PN-S reachability problem by the construction above.
4. For any rule application $(R_i) = $!P in Π, with conclusion $\Gamma_0 = \Xi_j$ and one premise Γ_1, if π denotes the index set of conclusion sequents of the !P rules occurring in Π below (R_i), by hypothesis $j \notin \pi$, and $N(\Gamma)$, $\{M(\Gamma_0, \pi)\}$, $\{M(\Gamma_1, \pi \uplus \{j\})\}$ satisfies the PN-S reachability problem by the construction above.

The composition of these reachability results along the depth of Π yields a promenade with initial marking $M(\Gamma, \emptyset)$ and all final markings $\emptyset$, thus, $N(\Gamma)$, $\{M(\Gamma, \emptyset)\}$, $\{\emptyset\}$ satisfies the PN-S reachability problem.

Secondly, assume $N(\Gamma)$, $\{M(\Gamma, \emptyset)\}$, $\{\emptyset\}$ satisfies the PN-S reachability problem. Then, there exists a promenade P on $N(\Gamma)$, where vertices of indegree 0 are labelled with $M(\Gamma, \emptyset)$ and vertices of outdegree 0 are labelled with $\emptyset$. Since $N(\Gamma)$ is a PN-S, i.e. has no join transition, we can assume without loss of generality that P has only one node of indegree 0, i.e. P is a tree. it is easy to see that any path in the tree P can be decomposed in sequences $(N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$, where only transfer transitions occur between the lock and the unlock transitions. All the transitions in such a sequence involve the same rule (R_i) applied on the same formula f_i. We say that such a sequence $s = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$ is *faithful* if the firing of the intermediate transfer (if applicable) transitions remove *all* tokens from M_i. Then, two cases arise:

1. In any path of P, any sequence $s = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_j)$ is faithful. In that case, the MELL sequents encoded by the markings M_i and the rules (R_i) induce naturally a proof Π of Γ, and $\vdash \Gamma$.
2. There exists a path c in P and an unfaithful sequence $s_i = (N(\Gamma), M_i) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_{i+1})$ in this path. Assume without loss of generality that s_i is the first unfaithful sequence in c, and denote by (R_i), Δ_i, π the corresponding rule, conclusion sequent underlying set and index set. Since s_i is the first unfaithful sequence in c, there exists $f \in \Delta$ such that $M_i(p^f_{\Delta,\pi}) > 0$ and $M_{i+1}(p^f_{\Delta,\pi}) > 0$. Since all paths reach the empty single marking by hypothesis, there exists a path $c' = s_1 \cdots s_k$ such that the sequence s_i is in $c \cap c'$, and the place $p^f_{\Delta,\pi}$ is emptied by some transition in the sequence $s_j = (N(\Gamma), M_j) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M_{j+1})$, with $j > i$, in c'. Then, the conclusion sequent underlying set and index set corresponding to s_j are $\Delta_j = \Delta_i$ and π, while the rule (R_j) may differ from (R_i). In other words, a token corresponding to a formula f is left behind during the simulation of the application of (R_i) on ζ_i with underlying set Δ_i and index set π, and inherited later on when simulating a rule (R_j) on ζ_j with the same underlying set $\Delta_j = \Delta_i$ and the same index set π.
Consider the subpath $s_i \cdots s_j$ of c', with the corresponding sequence of rules $(R_i) \cdots (R_j)$ and underlying sets $\Delta_i \cdots \Delta_j$. Note that for $i \leq k < j$, the index set corresponding to s_k is also π, by construction.

Therefore, none of the rules (R_k) for $i \leq k < j$ may be a !P rule, and, by Lemma 1.7, there exist sequences $s'_1 \cdots s'_j$ with $s'_k = (N(\Gamma), M'_k) \rightarrow_{t_{lock}} \cdots \rightarrow_{t_{unlock}} (N(\Gamma), M'_{k+1})$ such that:

(a) for any $i \leq k < j$, the rule corresponding to s'_k is (R_k),

(b) for any $i \leq k < j$, the conclusion sequent underlying set corresponding to s'_k is $\Delta_k \cup \{f\}$,

(c) for any $i \leq k < j$, s'_k is derived from s_k by changing the necessary indexes of places and transitions, and adding the necessary transfer transitions such that

- $M'_{k+1}(p^f_{\Delta_k \cup \{f\},\pi}) = 0$ if $\Delta_{k+1} \cup \{f\} \neq \Delta_k \cup \{f\}$, and $M'_{k+1}(p^f_{\Delta_{k+1} \cup \{f\},\pi}) > 0$ (i.e. no token of $p^f_{\Delta_k \cup \{f\},\pi}$ is left behind).
- for any $i \leq k \leq j$, for any $g \neq f$, $M'_k(p^g_{\Delta_k \cup \{f\},\pi}) = M_k(p^g_{\Delta_k,\pi})$.
- for any lock transition t being a split transition (i.e. corresponding to a $\otimes$ rule), the single marking produced by the firing of t which is not in c' remains unchanged.

In words, the sequences $s'_1 \cdots s'_j$ simulate the same rules as the sequences $s_1 \cdots s_j$, the same way, with the additional feature that they transfer inductively f from the conclusion sequent to the premise sequent.

The conditions above ensure that replacing the sequences $s_i \cdots s_j$ by the sequences $s'_i \cdots s'_j$ in P yield a graph P' that is also a promenade on $N(\Gamma)$, with (unique) vertex of indegree 0 labelled with $M(\Gamma, \emptyset)$ and vertices of outdegree labelled with $\emptyset$. Moreover, P' has strictly less unfaithful sequences than P. Therefore, by induction on the number of unfaithful sequences in P, we can conclude that there exists a promenade P'' on $N(\Gamma)$, with only faithful sequences and with (unique) vertex of indegree 0 labelled with $M(\Gamma, \emptyset)$ and vertices of outdegree labelled with $\emptyset$. Thus, $\vdash \Gamma$.

□

2. Vector Addition Systems with States and Split/Join Transitions

2.1. Definition

Definition 2.1. Vector Addition Systems with States and Split/Join Transitions.

A Vector Addition Systems with States and Split/Join transitions (VASS-SJ) is a 4-tuple $S = (G, T, m, v)$, where:

- $G = (Q, A)$ is a finite directed graph,
- $T \subseteq A \cup A^2$ is a set of *transitions*,
- $m \geq 1$ is a natural number,
- $v : T \rightarrow \mathbb{Z} \cup \{s\} \cup \{j\}$ is a function, such that
  1. $v(t) \in \mathbb{Z}^m$ if and only if $t \in A$,
  2. $v(t) = s$ if and only if $t = (a_1, a_2)$ and a_1 and a_2 share the same origin,
  3. $v(t) = j$ if and only if $t = (a_1, a_2)$ and a_1 and a_2 share the same destination,

The vertices of G are called its *states*. A transition t with $v(t) \in \mathbb{Z}^m$ is called a *regular* transition, a transition t with $v(t) = s$ is called a *split* transition, and a transition t with $v(t) = j$ is called a *join* transition.

A VASS-SJ with only regular and split transitions is called a *Vector Addition Systems with States and Split Transitions* (VASS-S), and a VASS-SJ with only regular and join transitions is called a *Vector Addition Systems with States and Join Transitions* (VASS-J).

Definition 2.2. Configuration of a VASS-SJ.

Let $S = (G, m, v)$ be a VASS-SJ. A *single configuration* of G is a 2-tuple $c = (q, x)$, where $q \in Q$ is a *state* and $x \in \mathbb{N}^m$ is a *value*. A single configuration c is *positive* if and only if, for all $0 \leq i \leq m$, $c_i \geq 0$. A *configuration* of S is a multiset $\mathcal{C}$ of single configurations of S. A configuration is *positive* if and only if all its single configurations are positive.

2.2. Execution Semantics of a VASS-SJ

Definition 2.3. Firing of a Transition.

Let $S = (G, T, m, v)$ be a VASS-SJ, and $\mathcal{C}$ be a configuration of S. Let t be a transition of S. The *firing* of t in $(S, \mathcal{C})$ is the relation:

$$(S, \mathcal{C}) \to_t (S, \mathcal{C}'),$$

where

- if $v(t) \in \mathbb{Z}^m$, $t = (q_0, q_1)$, there exists $c = (q_0, x_0) \in \mathcal{C}$, and $\mathcal{C}' = \mathcal{C} \setminus \{c\} \uplus \{(q_1, x_0 + v(t))\}$.
- if $v(t) = s$, $t = ((q_0, q_1), (q_0, q_1'))$, there exists $c_0 = (q_0, x_0) \in \mathcal{C}$, and $\mathcal{C}' = \mathcal{C} \setminus \{c\} \uplus \{(q_1, x_1)\} \uplus \{(q_1', x_1')\}$, with $x_0 = x_1 + x_1'$. Moreover, for any $1 \leq i \leq m$, $|(x_0)_i| = |(x_1)_i| + |(x_1')_i|$.
- if $v(t) = j$, $t = ((q_0, q_1), (q_0', q_1))$, there exist $c_0 = (q_0, x_0) \in \mathcal{C}$ and $c_0' = (q_0', x_0') \in \mathcal{C}$, and $\mathcal{C}' = \mathcal{C} \setminus \{c_0\} \setminus \{c_0'\} \uplus \{(q_1, x_0 + x_0')\}$.

Definition 2.4. Promenade.

Let $S = (G, T, m, v)$ be a VASS-SJ. A promenade on S is a labelled acyclic directed finite graph with in and out-degree at most two such that:

- each vertex is labelled with a single configuration on S,
- each edge is labelled with a transition of S,
- for any vertex, the (possibly two) ingoing edges have the same label, as well as the (possibly two) outgoing edges,
- for any vertex v labelled with c, with indegree two, with parent nodes v_1 labelled with c_1 and v_2 labelled with c_2, the ingoing edges are labelled with a join transition t, and we have:

$$(S, \{c_1\} \uplus \{c_2\}) \to_t (S, \{c\}),$$

- for any vertex v' labelled with c', with indegree one, with parent node v labelled with c with outdegree one, the ingoing edge is labelled with a regular transition t, and we have:

$$(S, \{c\}) \to_t (S, \{c'\}),$$

- for any vertex v labelled with c, with outdegree two, with child nodes v_1 labelled with c_1 and v_2 labelled with c_2, the outgoing edges are labelled with a split transition t, and we have:

$$(S, \{c\}) \to_t (S, \{c_1\} \uplus \{c_2\}).$$

A promenade P on S is *positive* if and only if all vertices are labelled with positive configurations.

Definition 2.5. (Positive) Reachability Problem.

Let S be a VASS-SJ, $\mathcal{C}_0$, $\mathcal{C}_1$ be two sets of single configurations on S. The reachability problem for S, $\mathcal{C}_0$, $\mathcal{C}_1$ is the following:
Does there exists a promenade P on S such that:

1. the set of labels of vertices of indegree 0 is $\mathcal{C}_0$, and
2. the set of labels of vertices of outdegree 0 is $\mathcal{C}_1$?

The positive reachability problem for S, $\mathcal{C}_0$, $\mathcal{C}_1$ is the following:
Does there exists a positive promenade P on S satisfying the same conditions as above?

The reachability problem for S, $\mathcal{C}_0$, $\mathcal{C}_1$ can easily be reformulated as follows:
Does there exist two configurations $\mathcal{C}'_0$, $\mathcal{C}'_1$ such that:

1. the underlying set of $\mathcal{C}'_0$ is $\mathcal{C}_0$,
2. the underlying set of $\mathcal{C}'_1$ is $\mathcal{C}_1$, and
3. there exists a finite sequence of transitions $t_0, \cdots, t_k$ of N such that

$$(S, \mathcal{C}'_0) \rightarrow_{t_0} \cdots \rightarrow_{t_k} (S, \mathcal{C}'_1).$$

The positive reachability problem can also be reformulated as above, with the condition that all configurations along the chain $(S, \mathcal{C}'_0) \rightarrow_{t_0} \cdots \rightarrow_{t_k} (S, \mathcal{C}'_1)$ are positive.

2.3. VASS-SJ and PN-SJ simulate each other

Proposition 2.6. 1. *The PN-SJ reachability problem and the positive VASS-SJ reachability problem reduce one to the other (via many-one reductions),*
2. *The PN-S reachability problem and the positive VASS-S reachability problem reduce one to the other (via many-one reductions),*
3. *The PN-J reachability problem and the positive VASS-J reachability problem reduce one to the other (via many-one reductions).*

Proof. Let us first prove PN-SJ reachability $\preceq_1$ positive VASS-SJ reachability.
Let $N = (P, T, W^-, W^+)$ be a PN-SJ, $\mathcal{M}_0$, $\mathcal{M}_1$ be two sets of single markings on N. Let us define the VASS-SJ $S = (G, T', m, v)$, with $G = (Q, A)$, as follows:

1. $m = |P|$, and we identify P with $[1 \cdots m]$,
2. to every regular transition r_i in T, we associate
   - a state q_i in Q,
   - a regular transition (q, q_i) with $v(q, q_i) = -W^-(r_i)$ (where we identify the function $W^-(t_i) : [1 \cdots m] \rightarrow \mathbb{N}$ with its vector in $\mathbb{Z}^m$), and
   - a regular transition (q_i, q) with $v(q_i, q) = W^+(r_i)$ in T'.
3. to every split transition s_j in T, we associate
   - three states o_j, $t1_j$ and $t2_j$ in Q,
   - a regular transition (q, o_j) with $v(q, o_j) = -W^-(s_j)$ in T',
   - a split transition $((o_j, t1_j), (o_j, t2_j))$ in T',
   - a regular transition $(t1_j, q)$ with $v(t1_j, q) = W^+_1(s_j)$, and
   - a regular transition $(t2_j, q)$ with $v(t2_j, q) = W^+_2(s_j)$ in T'.
4. to every join transition j_k in T, we associate
   - three states $o1_k$ and $o2_k$ and t_k in Q,
   - a regular transition $(q, o1_k)$ with $v(q, o1_k) = -W^-_1(j_k)$,
   - a regular transition $(q, o2_k)$ with $v(q, o2_k) = -W^-_2(j_k)$ in T',
   - a join transition $((o1_k, t_k), (o2_k, t_k))$ in T', and
   - a regular transition (t_k, q) with $v(t_k, q) = W^+(j_k)$ in T'.
5. Q is the disjoint union of $\{q\}$, and of the $\{q_i\}$, $\{o_j\}$, $\{t1_j\}$, $\{t2_j\}$, $\{o1_k\}$, $\{o2_k\}$ and $\{t_k\}$,defined above.

Let M be a single marking on N. To M, we associate the single configuration $c(M) = (q, (M(1) \cdots M(m)))$ of S. Then, N, $\mathcal{M}_0$, $\mathcal{M}_1$ satisfy the PN-SJ reachability problem if and only if S, $c(\mathcal{M}_0)$, $c(\mathcal{M}_1)$ satisfy the positive VASS-SJ reachability problem.
Let us now prove positive VASS-SJ reachability $\preceq_1$ PN-SJ reachability.
Let $S = (G, T, m, v)$, with $G = (Q, A)$ be a VASS-SJ, $\mathcal{C}_0$, $\mathcal{C}_1$ be two sets of single configurations on S. Let us define the PN-SJ $N = (P, T', W^-, W^+)$ as follows:

1. $P = Q \uplus \{1 \cdots m\}$,
2. to every regular transition $r_i = (o_i, t_i)$ in T, we associate a regular transition r_i in T', with
   - $W^-(q, r_i) = 0$ for $q \in Q \setminus \{o_i\}$,
   - $W^-(o_i, r_i) = 1$,
   - $W^-(k, r_i) = -v(r_i)_k$ for $k \in [1 \cdots m]$ and $v(r_i)_k < 0$, $W^-(k, r_i) = 0$ otherwise,
   - $W^+(q, r_i) = 0$ for $q \in Q \setminus \{t_i\}$,
   - $W^+(t_i, r_i) = 1$, and
   - $W^+(k, r_i) = v(r_i)_k$ for $k \in [1 \cdots m]$ and $v(r_i)_k \geq 0$, $W^+(k, r_i) = 0$ otherwise.
3. to every split transition $s_i = ((o_i, t1_i), (o_i, t2_i))$ in T, we associate a split transition s_i in T', with
   - $W^-(q, r_i) = 0$ for $q \in Q \setminus \{o_i\}$,
   - $W^-(o_i, r_i) = 1$,
   - $W^-(k, r_i) = 0$ for $k \in [1 \cdots m]$,
   - $W_1^+(q, r_i) = 0$ for $q \in Q \setminus \{t1_i\}$,
   - $W_1^+(t1_i, r_i) = 1$,
   - $W_1^+(k, r_i) = 0$ for $k \in [1 \cdots m]$,
   - $W_2^+(q, r_i) = 0$ for $q \in Q \setminus \{t2_i\}$,
   - $W_2^+(t2_i, r_i) = 1$, and
   - $W_2^+(k, r_i) = 0$ for $k \in [1 \cdots m]$.
4. to every join transition $j_i = ((o1_i, t_i), (o2_i, t_i))$ in T, we associate a join transition j_i in T', with
   - $W_1^-(q, r_i) = 0$ for $q \in Q \setminus \{o1_i\}$,
   - $W_1^-(o1_i, r_i) = 1$,
   - $W_1^-(k, r_i) = 0$ for $k \in [1 \cdots m]$,
   - $W_2^-(q, r_i) = 0$ for $q \in Q \setminus \{o2_i\}$,
   - $W_2^-(o2_i, r_i) = 1$,
   - $W_2^-(k, r_i) = 0$ for $k \in [1 \cdots m]$,
   - $W^+(q, j_i) = 0$ for $q \in Q \setminus \{t_i\}$,
   - $W^+(t_i, j_i) = 1$,
   - and $W^+(k, j_i) = 0$ for $k \in [1 \cdots m]$.

Let $c = (q, x)$ be a single configuration of S. To c, we associate the single marking $M(c)$ of S, where $M(c)(q') = 0$ if $q' \in Q \setminus \{q\}$, $M(c)(q) = 1$, and $M(c)(k) = x_k$ for $1 \leq k \leq m$. Then, S, $\mathcal{C}_0$, $\mathcal{C}_1$ satisfy the positive VASS-SJ reachability problem if and only if N, $M(\mathcal{C}_0)$, $M(\mathcal{C}_1)$ satisfy the PN-SJ reachability problem.
The same reductions apply for proving the other parts of the proposition.

□

3. Karp and Miller Graph

3.1. Definitions

Definition 3.1. Generalized configurations.

For $m \geq 1$, we consider the order relation $\leq$ on $(\mathbb{N} \cup \infty)^m$ defined as follows:

- for $m = 1$, $x \leq y$ if and only if $x \leq y \in \mathbb{N}$, or $y = \infty$, and

- for $m \geq 1$, $x \leq y$ if and only if, for all $i = 1 \cdots m$, $x_i \leq y_i$.

Given a VASS-SJ $S = (G, T, m, v)$, with $G = (Q, A)$, a *generalized single configuration* of S is a 2-tuple $g = (q, x)$, where $q \in Q$ is a *state* and $x \in (\mathbb{N} \cup \infty)^m$ is a *generalized value*. A *generalized configuration* is a finite set $\mathcal{G}$ of generalized single configurations of S, together with a multiplicity function $M_{\mathcal{G}} : \mathcal{G} \to \mathbb{N} \cup \infty$, i.e. a finite multiset whose elements may have finite or infinite multiplicity. For a generalized configuration $\mathcal{G}$ and $g \in \mathcal{C}$, we write $\mathcal{G} \setminus g$ for the generalized configuration $\mathcal{G}'$ such that:

1. for all $g' \neq g$, $g' \in \mathcal{G}'$ if and only if $g' \in \mathcal{G}$. In that case $M_{\mathcal{G}}(g') = M_{\mathcal{G}'}(g')$, and
2. if $M_{\mathcal{G}}(g) > 1$, $g \in \mathcal{G}'$ with $M_{\mathcal{G}'}(g) = M_{\mathcal{G}}(g) - 1$ (where, of course, $\infty - 1 = \infty$),
3. if $M_{\mathcal{G}}(g) = 1$, $g \notin \mathcal{G}'$.

For a generalized configuration $\mathcal{G}$, $k \in \mathbb{N} \uplus \infty$, and a generalized single configuration g, we write $\mathcal{G} \uplus^k c$ for the generalized configuration $\mathcal{G}'$ such that:

1. for all $g' \neq g$, $g' \in \mathcal{G}'$ if and only if $g' \in \mathcal{G}$. In that case $M_{\mathcal{G}}(g') = M_{\mathcal{G}'}(g')$, and
2. if $g \in \mathcal{G}$, $g \in \mathcal{G}'$ with $M_{\mathcal{G}'}(g) = M_{\mathcal{G}}(g) + k$,
3. if $g \notin \mathcal{G}$, $g \in \mathcal{G}'$ with $M_{\mathcal{G}'}(g) = k$.

And, for two generalized configurations $\mathcal{G}, \mathcal{G}'$, with $\mathcal{G}' = \{g'_1, \cdots, g'_k\}$,

$$\mathcal{G} \uplus \mathcal{G}' = \mathcal{G} \uplus^{M_{\mathcal{G}'}(g'_1)} g'_1 \cdots \uplus^{M_{\mathcal{G}'}(g'_k)} g'_k, \text{ and}$$

$$\mathcal{G} \subseteq \mathcal{G}' \text{ iff } \forall g \in \mathcal{G}, g \in \mathcal{G}' \text{ with } M_{\mathcal{G}}(g) \leq M_{\mathcal{G}'}(g).$$

For a set $\mathcal{D}$ of generalized single configurations, we write also $\mathcal{D}$ for the generalized configuration with underlying set $\mathcal{D}$, where all elements have multiplicity 1.
We consider the order relation $\preceq$ on generalized configurations defined as follows: $\mathcal{G} \preceq \mathcal{G}'$ if and only if $\mathcal{G}'$ can be derived in a finite number of steps from $\mathcal{G}$ by:

- increasing some coordinate of an element $g \in \mathcal{G}$ (while keeping other coordinates invariant), or
- performing some union $\mathcal{G} \uplus^k g$ for some generalized single configuration g, with $k \in \mathbb{N} \uplus \infty$.

Moreover, we assume for the sake of simplicity that the reduction steps consisting in increasing some coordinate are not redundant (i.e. the same coordinate of the same single configuration is increased at most once in the derivation), that the reduction steps consisting in adding some single configuration are not redundant either (i.e. a single configuration is added at most once in the derivation), and that the former steps occur before the latter in the derivation. Note that this assumption does not change the definition, but allows to simplify the presentation of the liftings below.

Definition 3.2. Liftings.

Let $S = (G, T, m, v)$, with $G = (Q, A)$ be a VASS-SJ. Let $\mathcal{T}$ be a connected, directed acyclic graph with vertices labelled with generalized configurations on S. Let d_n be a vertex in $\mathcal{T}$ labelled with $\mathcal{G}_n$. Let $\mathcal{G}_i$ be a generalized configuration on S. We write $\mathcal{G}_i \ll \mathcal{G}_n$ if and only if $\mathcal{G}_i \prec \mathcal{G}_n$ and there exits a path $d_i, \cdots, d_n$ in $\mathcal{T}$, where $d_i \neq d_n$ is labelled with $\mathcal{G}_i$. Assume $\mathcal{G}_i \ll_{d_n} \mathcal{G}_n$ and let $\mathcal{G}_i = \mathcal{G}'_1 \to \cdots \to \mathcal{G}'_m = \mathcal{G}_n$ be one corresponding finite non-empty derivation.
The *single lifting* of (d_n, d_i) in $\mathcal{T}$, denoted as $\#(d_n, d_i)$ is the following operation. For all $k = 1 \cdots m - 1$,

- If, in the derivation step $\mathcal{G}'_k \to \mathcal{G}'_{k+1}$, the coordinate x_j of some $g = (l, (p, x)) \in \mathcal{G}'_k$ has strictly increased by a value a, replace it by ∞ in $\mathcal{G}'_{k+1}, \cdots, \mathcal{G}'_m$. The coordinate x_j is denoted as an *augmentation coordinate* of the lifting, and a is its *augmentation gap*.
- If, in the derivation step $\mathcal{G}'_k \to \mathcal{G}'_{k+1}$, a generalized single configuration g has been added, with multiplicity $a \in \mathbb{N}$, give it multiplicity ∞ in $\mathcal{G}'_{k+1}, \cdots, \mathcal{G}'_m$. The configuration g is denoted as an *augmentation configuration* of the lifting, and a is its *augmentation gap*.

- Label d_n with the resulting generalized configuration $\mathcal{G}_n$.

The *augmentation gap* of the lifting is the minimal non-zero augmentation gap of the augmentation coordinates and of the augmentation configurations of the lifting.
Note that two single liftings commute: $\forall i, j \leq n, \#(\#(d_n, d_i), d_j) = \#(\#(d_n, d_j), d_i)$. Now, let $J = \{j = j_1, \cdots, j_k$ such that $\mathcal{G}_j \ll \mathcal{G}_n\}$, let $D^0 = d_n$ and for $l = 1 \cdots k$, $D^l = \#(D^{l-1}, d_{j_l})$. Then, the *lifting* of d_n, is $\#(d_n) = D^k$. Note that, when $J = \emptyset$, $k = 0$ and $\#(d_n) = d_n$.

Definition 3.3. Karp and Miller Graph.

Let $S = (G, T, m, v)$, with $G = (Q, A)$ be a VASS-SJ and $\mathcal{D}$ be a set of single configurations. The *Karp and Miller graph* $\mathcal{T} = (V, E)$ on $(S, \mathcal{D})$ is a labelled directed graph constructed inductively as follows, where vertices are labelled with generalized configurations, and edges with transitions of S or (null).

1. $\mathcal{T}$ has exactly one vertex σ_0 of indegree 0, labelled with $\emptyset$
2. For every $c \in \mathcal{D}$, let σ_c be a new vertex labelled with $\{c\}$, where c has multiplicity 1. Let (σ_0, σ_c) be a new edge, labelled with (null).
3. Let σ_i labelled with $\mathcal{G}_i$ be a vertex of $\mathcal{T}$. If there exists a path $\sigma_0, \cdots, \sigma_i$ i $\mathcal{T}$ and $\sigma_j \neq \sigma_i$ labelled with the same $\mathcal{G}_i$ in this path, σ_i has outdegree 0. Otherwise,
4. Let σ_i labelled with $\mathcal{G}_i$ be a vertex of $\mathcal{T}$. Let $(p_i, x_i) \in \mathcal{G}_i$, $t = (p_i, p_{i+1}) \in T$ be a regular transition such that $x_{i+1} = x_i + v(t) \geq 0$. Then, let σ_{i+1} be a new vertex in V labelled with $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \uplus^1 (p_{i+1}, x_{i+1})$, and (σ_i, σ_{i+1}) be a new edge in E labelled with t. Mark one occurrence of $(p_i, x_i) \in \mathcal{G}_i$ as the *origin of* (σ_i, σ_{i+1}), one occurrence of (p_{i+1}, x_{i+1}) in $\mathcal{G}_{i+1}$ as the *destination of* (σ_i, σ_{i+1}), and perform the lifting $\#(\sigma_{i+1})$ in $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$ (where we assume that the lifting preserves the marking).
5. Let σ_i labelled with $\mathcal{G}_i$ be a vertex of $\mathcal{T}$. Let $(p_i, x_i) \in \mathcal{G}_i$, $t = ((p_i, p_{i+1}), (p_i, p'_{i+1})) \in T$ be a split transition. Let $x_{i+1} + x'_{i+1} = x_i$, such that, for $j = 1 \cdots m$, $x_{ij} = \infty \Rightarrow x_{i+1_j}, x'_{i+1_j} = \infty$ $x'_{i+1_j} \in \{0, \infty\}$?. Then, let σ_{i+1} be a new vertex in V labelled with $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \uplus^1 (p_{i+1}, x_{i+1}) \uplus^1 (p'_{i+1}, x'_{i+1})$, (σ_i, σ_{i+1}) be a new edge in E labelled with t. Mark one occurrence of $(p_i, x_i) \in \mathcal{G}_i$ as the *origin of* (σ_i, σ_{i+1}), one occurrence of (p_{i+1}, x_{i+1}) in $\mathcal{G}_{i+1}$ as a *destination of* (σ_i, σ_{i+1}), one occurrence of (p'_{i+1}, x'_{i+1}) in $\mathcal{G}_{i+1}$ as a *destination of* (σ_i, σ_{i+1}), and perform the lifting $\#(\sigma_{i+1})$ in $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$ (where we assume that the lifting preserves the marking).
6. Let σ_i labelled with $\mathcal{G}_i$ be a vertex of $\mathcal{T}$. Let $(p_i, x_i) \in \mathcal{G}_i$, $(p'_i, x'_i) \in \mathcal{G}_i$, $t = ((p_i, p_{i+1}), (p'_i, p_{i+1})) \in T$ be a join transition. Let $x_{i+1} = x_i + x'_i$. Then, let σ_{i+1} be a new vertex in V labelled with $\mathcal{G}_{i+1} = \mathcal{G}_i \setminus (p_i, x_i) \setminus (p'_i, x'_i) \uplus^1 (p_{i+1}, x_{i+1})$, (σ_i, σ_{i+1}) be a new edge in E labelled with t. Mark one occurrence of $(p_i, x_i) \in \mathcal{G}_i$ as an *origin of* (σ_i, σ_{i+1}), one occurrence of $(p'_i, x'_i) \in \mathcal{G}_i$ as an *origin of* (σ_i, σ_{i+1}), one occurrence of (p_{i+1}, x_{i+1}) in $\mathcal{G}_{i+1}$ as the *destination of* (σ_i, σ_{i+1}), and perform the lifting $\#(\sigma_{i+1})$ in $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$ (where we assume that the lifting preserves the marking).
7. Let σ_i labelled with $\mathcal{G}_i$, σ_j labelled with $\mathcal{G}_j$ be two vertices of $\mathcal{T}$. Let $(p_i, x_i) \in \mathcal{G}_i$, $(p_j, x_j) \in \mathcal{G}_j$, $t = ((p_i, p_{i+1}), (p_j, p_{i+1})) \in T$ be a join transition. Let $x_{i+1} = x_i + x_j$. Then, let σ_{i+1} be a new vertex in V labelled with $\mathcal{G}_{i+1} = (\mathcal{G}_i \setminus (p_i, x_i)) \uplus (\mathcal{G}_j \setminus (p_j, x_j)) \uplus^1 (p_{i+1}, x_{i+1})$, (σ_i, σ_{i+1}) and (σ_j, σ_{i+1}) be two new edges in E labelled with t. Mark one occurrence of $(p_i, x_i) \in \mathcal{G}_i$ as an *origin of* (σ_i, σ_{i+1}), one occurrence of $(p_j, x_j) \in \mathcal{G}_j$ as an *origin of* (σ_j, σ_{i+1}), one occurrence of (p_{i+1}, x_{i+1}) in $\mathcal{G}_{i+1}$ as the *destination of* (σ_i, σ_{i+1}) *and of* (σ_j, σ_{i+1}), and perform the lifting $\#(\sigma_{i+1})$ in $(V \uplus \sigma_{i+1}, E \uplus (\sigma_i, \sigma_{i+1}))$ (where we assume that the lifting preserves the marking).

The inductive construction above halts when no new vertex can be added with these rules. Note that, by construction, $\mathcal{T}$ is connected and acyclic.

3.2. First results

Proposition 3.4. *The following statements are true:*

1. *Let $u_n \in (\mathbb{N} \uplus \infty)^m$, $n \in \mathbb{N}$, be an infinite sequence of m-tuples for some $m \in \mathbb{N}$. Then, there exists an infinite sub-sequence u'_n, $n \in \mathbb{N}$ of u_n that is increasing for the order relation $\leq$.*
2. *Let $\mathcal{G}_n$, $n \in \mathbb{N}$, be an infinite sequence of generalized configurations. Then, there exists an infinite sub-sequence $\mathcal{G}'_n$, $n \in \mathbb{N}$ of $\mathcal{G}_n$ that is increasing for the order relation $\preceq$.*

Proof.

1. Let $u_n \in (\mathbb{N} \uplus \infty)^m$, $n \in \mathbb{N}$, be an infinite sequence of m-tuples for some $m \in \mathbb{N}$. Then, two cases arise:
   - The sequence u_n is bounded by some $u \in \mathbb{N}^m$. In that case, there exists $u' \leq u$ such that u' has infinitely many occurrences in u_n. The restriction of u_n to these occurrences is an infinite sequence that is (non-strictly) increasing.
   - The sequence u_n is unbounded. By induction on m:
     (a) if $m = 1$, if ∞ has infinitely many occurrences in u_n: the restriction of u_n to these occurrences is an infinite sequence that is (non-strictly) increasing. if ∞ has only finitely many occurrences in u_n: there exists an infinite unbounded sub-sequence $u'_n \in \mathbb{N}$, $n \in \mathbb{N}$ with values in $\mathbb{N}$. Then,
       i. if the sequence u'_n is bounded by some $u \in \mathbb{N}^m$, it admits as above an infinite sequence that is (non-strictly) increasing, and
       ii. if the sequence u'_n is unbounded, it admits an infinite sub-sequence that is (strictly) increasing.
     (b) Assume the result holds for $m-1$. Consider by induction an infinite subsequence u'_n, $n \in \mathbb{N}$ of u_n such that its projection on the first $m-1$ coordinates is increasing, and apply the result for $m = 1$ on the last coordinate of u'_n.
2. Let now $\mathcal{G}_n$, $n \in \mathbb{N}$, be an infinite sequence of generalized configurations. Let $\mathcal{G}'_n$, $n \in \mathbb{N}$, be an infinite sub-sequence of $\mathcal{G}_n$, such that the size of the underlying sets of $\mathcal{G}'_n$, is an infinite sequence of integer values that is increasing. Let us show that, for any $\mathcal{G}'_i$ in the sequence $\mathcal{G}'_n$, there exists $\mathcal{G}'_j$, $i < j$ such that $\mathcal{G}'_i \preceq \mathcal{G}'_j$. For $q \in Q$, let $v(q) \in \mathbb{N}^{|Q|}$ be the unary encoding of q. Let $\mathcal{G}'_i = \{(q^i_1, x^i_1), \cdots, (q^i_l, x^i_l)\}$ in the sequence, and, for $j \geq i$, let $\mathcal{G}'_j = \{(q^j_1, x^j_1), \cdots, (q^j_{l'}, x^j_{l'})\}$ with, since the size of the sets increase, $l' \geq l$. Consider now only the l first elements of $\mathcal{G}'_j$ in $v(\mathcal{G}'_j) = (v(q^j_1), x^j_1, M(\mathcal{G}'_j)(q^j_1, x^j_1), \cdots, v(q^j_l), x^j_l, M(\mathcal{G}'_j)(q^j_l, x^j_l)) \in (\mathbb{N} \uplus \infty)^{l.(|Q|+m+1)}$.
   Consider now the sequence $v(\mathcal{G}'_j)$, $j \geq i$. Then, by Proposition 3.4, 1), there exists an infinite sub-sequence $v(\mathcal{G}''_j)$, $j \geq i$ that is increasing for the order relation $\leq$ on $(\mathbb{N} \uplus \infty)^{l.(|Q|+m+1)}$, and let $\mathcal{G}''_n$, $n \in \mathbb{N}$ be the corresponding sequence of generalized configurations. Then, by construction of $v(\mathcal{G}''_j)$, $\mathcal{G}'_i \preceq \mathcal{G}''$ for any $\mathcal{G}'' = \mathcal{G}''_j$ in the sequence $\mathcal{G}''_n$. It follows that, for any $\mathcal{G}'_i$ in the sequence $\mathcal{G}'_n$, there exists $\mathcal{G}'_j$, $i < j$ such that $\mathcal{G}'_i \preceq \mathcal{G}'_j$. Therefore, there exists an infinite sub-sequence of $\mathcal{G}'_n$, $n \in \mathbb{N}$ that is increasing for the order relation $\preceq$.

□

Theorem 3.5. *Let $S = (G, T, m, v)$, with $G = (Q, A)$ be a VASS-SJ and $\mathcal{D}$ be a finite set of single configurations. The Karp and Miller graph $\mathcal{T}$ on $(S, \mathcal{D})$ is finite, and can be effectively constructed.*

Proof. Since $\mathcal{T}$ is a connected, directed acyclic graph, it is finite if and only if its depth-first search tree is finite. By construction, every node in $\mathcal{T}$ has a finite number of sons. Assume $\mathcal{T}$ is infinite: then, by Koenig's Lemma, its depth-first search tree has an infinite branch. Denote by $D = \mathcal{G}_n$, $n \in \mathbb{N}$ the corresponding infinite derivation sequence of generalized configurations. By Proposition 3.4, 2), let $\mathcal{G}'_n$, $n \in \mathbb{N}$ be an infinite sub-sequence of $\mathcal{G}_n$ that is increasing for the order relation $\preceq$.
Let $\mathcal{G}'_i$ be one element in the sequence $\mathcal{G}'_n$, and define $\mathcal{G}'_i = I(\mathcal{G}'_i) \uplus F(\mathcal{G}'_i)$, where:

- $I(\mathcal{G}'_i) = \{g \in \mathcal{G}'_i : M_{\mathcal{G}'_i}(g) = \infty\}$, with $M_{I(\mathcal{G}'_i)} = M_{\mathcal{G}'_i}(g)$ for all g, and
- $F(\mathcal{G}'_i) = \{g \in \mathcal{G}'_i : M_{\mathcal{G}'_i}(g) \in \mathbb{N}\}$, with $M_{I(\mathcal{G}'_i)} = M_{\mathcal{G}'_i}(g)$ for all g.

Let $Q = \{q_1, \cdots, q_{|Q|}\}$, let $\mathcal{G}'_i = \{(q^i_1, x^i_1), \cdots, (q^i_l, x^i_l)\}$, $l \in \mathbb{N}$, and define now:

- for $j = 1, \cdots, m$, for $q \in Q$, $v_{j,q}(\mathcal{G}'_i) = (\sum_{(q,x)\in I(\mathcal{G}'_i):\ x_j\neq\infty} x_j, \sum_{(q,x)\in F(\mathcal{G}'_i):\ x_j\neq\infty} x_j.M_{\mathcal{G}'_i}(q,x)) \in \mathbb{N}^2$, and
- $v(\mathcal{G}'_i) = \big(v_{1,q_1}(\mathcal{G}'_i), v_{2,q_1}(\mathcal{G}'_i), \cdots, v_{m,q_1}(\mathcal{G}'_i), v_{1,q_2}(\mathcal{G}'_i), \cdots, v_{m,q_2}(\mathcal{G}'_i), \cdots, v_{m,q_{|Q|}}(\mathcal{G}'_i)\big) \in \mathbb{N}^{2.m.|Q|}$:

Consider now the infinite sequence $v(\mathcal{G}'_n)$, $n \in \mathbb{N}$. Then, by Proposition 3.4, 1), let $v(\mathcal{G}''_n)$, $n \in \mathbb{N}$ be an infinite sub-sequence of $v(\mathcal{G}'_n)$ that is increasing for the order relation $\leq$ on $\mathbb{N}^{2.m.|Q|}$, and let $\mathcal{G}''_n$, $n \in \mathbb{N}$ be the corresponding infinite sequence of generalized configurations.
Two cases arise:

1. $v(\mathcal{G}''_n)$ is bounded: there exists $i \in \mathbb{N}$ such that, for all $j > i$, $v(\mathcal{G}''_i) = v(\mathcal{G}''_j)$. Since the number of possible generalized configurations $\mathcal{G}''$ with $v(\mathcal{G}'') = v(\mathcal{G}''_i)$ is bounded, it follows that there exists an infinite sub-sequence of $\mathcal{G}''_n$ of identical generalized configurations, which is contradictory with the construction of the Karp and Miller tree (case 3) of the construction in Definition 3.3).
2. $v(\mathcal{G}''_n)$ is not bounded: there exists yet again an infinite subsequence $v(\mathcal{G}^{(3)}_n)$, $n \in \mathbb{N}$, which is strictly increasing for the order relation $\leq$ on $\mathbb{N}^{2.m.|Q|}$. Consider the corresponding infinite sequence $\mathcal{G}^{(3)}_n$, $n \in \mathbb{N}$ of generalized configurations. For any $i < j \in \mathbb{N}$, $v(\mathcal{G}^{(3)}_i) < v(\mathcal{G}^{(3)}_j)$, and $\mathcal{G}^{(3)}_i \preceq \mathcal{G}^{(3)}_j$. Then, $F(\mathcal{G}^{(3)}_i) = F(\mathcal{G}^{(3)}_j)$, otherwise, by construction, a single lifting needs to be performed on the finite part of $\mathcal{G}^{(3)}_j$, which would make $v(\mathcal{G}^{(3)}_j)$ strictly decrease on its coordinates corresponding to $F(\mathcal{G}^{(3)}_j)$. Similarly, any derivation of $\mathcal{G}^{(3)}_i \preceq \mathcal{G}^{(3)}_j$ consists in adding at least one single configuration with infinite multiplicity (otherwise, any derivation consisting only in increasing values would trigger a lifting that would make v strictly decrease). Consider now the infinite sequence of single configuration with infinite multiplicity added in each of the derivations $\mathcal{G}^{(3)}{}_n \preceq \mathcal{G}^{(3)}{}_{n+1}$ for $n \in \mathbb{N}$. By Proposition 3.4, 1), it has an infinite sub-sequence that is increasing for the order $\leq$ on $(\mathbb{N} \uplus \infty)^m$. If this infinite sub-sequence is bounded, it contains yet one more infinite sub-sequence of identical single configurations, which contradicts the construction of the graph (case 3) of the construction in Definition 3.3); if it is unbounded, it contains yet one more infinite sub-sequence that is strictly increasing, which contradicts the fact that the $\mathcal{G}_i$ are lifted at each step in the derivation sequence (case 4), 5), 6) and 7) of the construction in Definition 3.3).

It follows by contradiction that the depth-first search tree of the Karp and Miller graph has no infinite branch, hence it is finite. The constructibility follows from its inductive definition. □

Theorem 3.6. *Let $S = (G, T, m, v)$, with $G = (Q, A)$ be a VASS-SJ and $\mathcal{D}$ be a finite set of single configurations. Let σ be a node of the Karp and Miller graph $\mathcal{T}$ on $(S, \mathcal{D})$ labelled with $\mathcal{G}^\sigma$.*
Let $\mathcal{D}_\sigma \subseteq \mathcal{D}$ such that, for all $c \in \mathcal{D}_\sigma$, there exists a path $\sigma_c, \cdots, \sigma$ in $\mathcal{T}$.
Then, for any $N \in \mathbb{N}$, there exists a promenade P^N_σ of S such that:

- *the set of labels of the vertices of P^N_σ of in-arity 0 is $\mathcal{D}_\sigma$,*
- *for all $g = (q, x) \in \mathcal{G}^\sigma$ of multiplicity $M_{\mathcal{G}^\sigma}(g) = k \in \mathbb{N}$, there exists a set $V(g) = \{v_1, \ldots, v_k\}$ of vertices of P^N_σ of out-arity 0, where, for any $v_i \in V(g)$, v_i is labelled with (q, t^i) where, for all $j = 1, \cdots, m$:*
  1. *$t^i_j \geq N$ if $x_j = \infty$, and*
  2. *$t^i_j = x_j$ otherwise,*
- *for all $g = (q, x) \in \mathcal{G}^\sigma$ of multiplicity $M_{\mathcal{G}^\sigma}(g) = \infty$, there exists a set $V(g) = \{v_1, \ldots, v_k\}$, $k \geq N$, of vertices of P^N_σ of out-arity 0, where, for any $v_i \in V(g)$, v_i is labelled with (q, t^i) where, for all $j = 1, \cdots, m$:*
  1. *$t^i_j \geq N$ if $x_j = \infty$, and*
  2. *$t^i_j = x_j$ otherwise,*
- *and the union of the sets $V(g)$ above is the set of vertices of P^N_σ of out-arity 0.*

Proof. For the sake of simplicity, we assume in the proof that all directed acyclic graphs are oriented from top to bottom.

Let $\Sigma(\sigma) = \{\sigma_1, \cdots, \sigma_t\}$ be the set of vertices in $\mathcal{T} \setminus \sigma_0$ (recall Definition 3.3: σ_0 is the only vertex of $\mathcal{T}$ of in-degree 0) above σ (i.e, such that, for $j = 1, \cdots, t$, there exists a path from σ_j to σ in $\mathcal{T}$). Assume also that, for $j = 1, \cdots, t$, σ_j is labelled with $\mathcal{G}^j$. Let $\mathcal{T}_\sigma$ be the restriction of $\mathcal{T}$ to $\Sigma(\sigma)$, as given in Figure 2 for the VASS-SJ S of Figure 1.

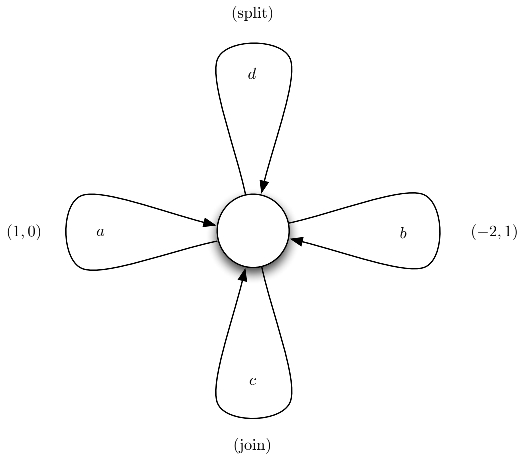

Figure 1: An example of VASS-SJ S, with only one state, two regular transitions a with $v(a) = (1, 0)$ and b with $v(b) = (-2, 1)$, a join transition c and a split transition d.

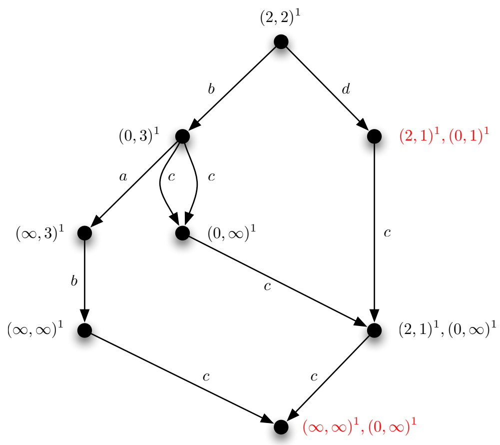

Figure 2: The graph $\mathcal{T}_\sigma$ for $(S, \{(2, 2)\}$, with one vertex σ labelled with $(\infty, \infty)^1, (0, \infty)^1$.

`Splitting generalized configurations into single ones:`

Denote now by $\mathcal{T}'_\sigma$ the directed acyclic graph defined inductively as follows, as in figure 3:

1. To σ, we associate a set $v(\sigma)$ of nodes of out-degree 0, where, for any $g \in \mathcal{G}^\sigma$, $v(\sigma, g) \in v(\sigma)$ is labelled with $g^{M_{\mathcal{G}^\sigma}(g)}$.

2. Let $v(\sigma_k) \in \mathcal{T'}_\sigma$, where $\sigma_k \in \mathcal{T}_\sigma$ has only one parent vertex $\sigma_i \in \mathcal{T}_\sigma$. Assume (σ_i, σ_k) is labelled with a transition t. Then, we let $v(\sigma_i)$ be a new set of nodes in $\mathcal{T'}_\sigma$, where, for $g_i \in \mathcal{G}^i$,
   - If g_i is not marked in $\mathcal{G}^i$ as the origin of (σ_i, σ_k), g_i yields $g_k \in \mathcal{G}^k$ by the (possibly empty) lifting of σ_k. Then, $v(\sigma_i, g_i)$ is a new node in $v(\sigma_i)$ labelled with $g_i^{M_{\mathcal{G}^i}(g_i)}$, and $(v(\sigma_i, g_i), v(\sigma_k, g_k))$ is a new edge labelled with (id), with $v(\sigma_k, g_k) \in v(\sigma_k)$.
   - if g_i is marked in $\mathcal{G}^i$ as the origin of (σ_i, σ_k), $v(\sigma_i, g_i)$ is a new node in $v(\sigma_i)$ labelled with $g_i^{M_{\mathcal{G}^i}(g_i)}$. Let $g_k \in \mathcal{G}^k$ marked as the destination of (σ_i, σ_k). Then, $(v(\sigma_i, g_i), v(\sigma_k, g_k))$ is a new edge labelled with t, with $v(\sigma_k, g_k) \in v(\sigma_k)$.
3. Let $v(\sigma_k) \in \mathcal{T'}_\sigma$, where $\sigma_k \in \mathcal{T}_\sigma$ has only two parent vertices $\sigma_i \in \mathcal{T}_\sigma$ and $\sigma_j \in \mathcal{T}_\sigma$. Assume (σ_i, σ_k) and (σ_j, σ_k) are labelled with the same transition t. Then, we let $v(\sigma_i)$ and $v(\sigma_j)$ be two new sets of nodes in $\mathcal{T'}_\sigma$, where, for $g_i \in \mathcal{G}^i$ (respectively $g_j \in \mathcal{G}^j$),
   - If g_i is not marked in $\mathcal{G}^i$ as the origin of (σ_i, σ_k), g_i yields $g_k \in \mathcal{G}^k$ by the (possibly empty) lifting of σ_k. Then, $v(\sigma_i, g_i)$ is a new node in $v(\sigma_i)$ labelled with $g_i^{M_{\mathcal{G}^i}(g_i)}$, and $(v(\sigma_i, g_i), v(\sigma_k, g_k))$ is a new edge labelled with (id), with $v(\sigma_k, g_k) \in v(\sigma_k)$.
   - if g_i is marked in $\mathcal{G}^i$ as the origin of (σ_i, σ_k), $v(\sigma_i, g_i)$ is a new node in $v(\sigma_i)$ labelled with $g_i^{M_{\mathcal{G}^i}(g_i)}$. Let $g_k \in \mathcal{G}^k$ marked as the destination of (σ_i, σ_k). Then, $(v(\sigma_i, g_i), v(\sigma_k, g_k))$ is a new edge labelled with t, with $v(\sigma_k, g_k) \in v(\sigma_k)$.

   (respectively,
   - If g_j is not marked in $\mathcal{G}^j$ as the origin of (σ_j, σ_k), g_j yields $g_k \in \mathcal{G}^k$ by the (possibly empty) lifting of σ_k. Then, $v(\sigma_j, g_j)$ is a new node in $v(\sigma_j)$ labelled with $g_j^{M_{\mathcal{G}^j}(g_j)}$, and $(v(\sigma_j, g_j), v(\sigma_k, g_k))$ is a new edge labelled with (id), with $v(\sigma_k, g_k) \in v(\sigma_k)$.
   - if g_j is marked in $\mathcal{G}^j$ as the origin of (σ_j, σ_k), $v(\sigma_j, g_j)$ is a new node in $v(\sigma_j)$ labelled with $g_j^{M_{\mathcal{G}^j}(g_j)}$. Let $g_k \in \mathcal{G}^k$ marked as the destination of (σ_j, σ_k). Then, $(v(\sigma_j, g_j), v(\sigma_k, g_k))$ is a new edge labelled with t, with $v(\sigma_k, g_k) \in v(\sigma_k)$.)

Note that, to a node $\sigma_i \in \mathcal{T}_\sigma$, we associate several sets $v(\sigma_i) \in \mathcal{T'}_\sigma$. The sets $v(\sigma_i) \in \mathcal{T'}_\sigma$ are a partition of $\mathcal{T'}_\sigma$, denoted as V_σ. Note also that the graph $\mathcal{T''}_\sigma$ obtained from $\mathcal{T'}_\sigma$ by merging all nodes of all sets $v(\sigma_i)$ is a tree, whose root is the node obtained by merging all nodes of $v(\sigma)$.

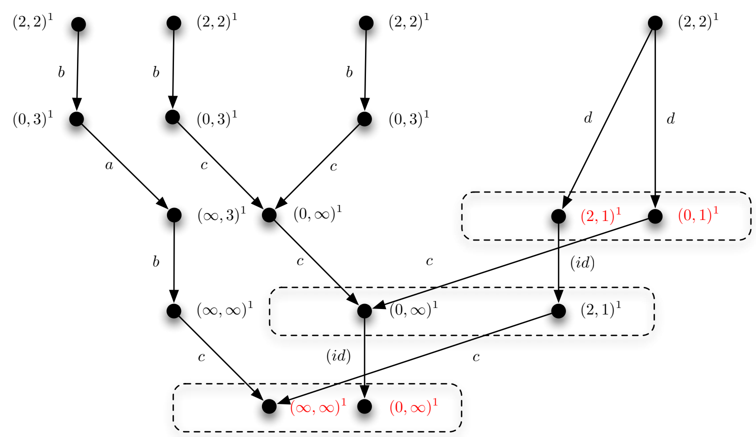

Figure 3: The graph $\mathcal{T'}_\sigma$ obtained from $\mathcal{T}_\sigma$. The dotted boxes are the sets $v(\sigma_i)$ of cardinality more than 2.

`The promenade` P_σ`:`

From now on, assume in first step that no generalized single configuration appears in $\mathcal{T}'_\sigma$ with multiplicity ∞. The case where ∞ multiplicities occur will be treated in a second step.
Under this aussmption, the graph $\mathcal{T}'_\sigma$ induces naturally a (non-necessarily positive) promenade P_σ of S as in Figure 4 (where, without loss of generality, we extend the notion of promenade with (id) transitions), together with a partition V'_σ of its nodes, by inductively firing from top to bottom the transitions labeling the edges of $\mathcal{T}'_\sigma$, starting from the sets $v(\sigma_i)$ with no in-going edge, corresponding to single configurations in $\mathcal{D}$. The only ambiguous case is when firing a split transition on a single configuration in P_σ corresponding to a single generalized configuration in $\mathcal{T}'_\sigma$ with ∞ coordinates. In this case, by convention, the corresponding finite coordinates z_i of the single configuration in P_σ are split into z'_i and z''_i with $|z'_i - z''_i| \leq 1$. For a set $v \in V_\sigma$, denote by $P(v)$ its image in P_σ by the construction above. Then, the image $P(V_\sigma)$ of V_σ by the inductive construction above is a partition of the vertices of P_σ. Note that, as for $\mathcal{T}'_\sigma$, the graph obtained from P_σ by merging all nodes that belong to the same set $P(v) \in P(V_\sigma)$ into a single node is a tree.

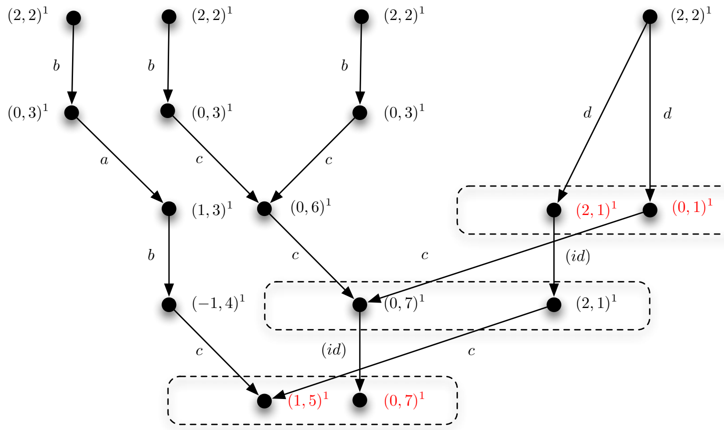

Figure 4: The promenade P_σ corresponding to the graph $\mathcal{T}'_\sigma$.

`Tree property:`

Let P be a promenade of S, Let V be a partition of the vertices of P. We say that (P, V) satisfies the tree property if and only if the graph obtained from P by contracting all nodes that belong to a same set $v \in V$ into a single node is a tree. Now, from P_σ and $P(V_\sigma)$, we will construct inductively a promenade that satisfies the theorem, together with a partition of its node, that satisfies the tree property.

`Repeating loops:`

Let P be a promenade of S, V be a partition of the vertices of P such that (P, V) satisfies the tree property. Let v_i above $v_j \in V$. Denote by c_i the configuration of S labeling v_i (i.e, c_i is the multiset of the labels of the nodes in v_i), and by c_j the configuration of S labeling v_j, and assume that c_j can be obtained from c_i by only modifying the values at some coordinates of some single configurations.
Then, the *k-repetition of the chain $v_i \cdots v_j$ in P*, for $k > 1$ is the promenade $P^k_{i,j}$ obtained from P by:

1. isolating the subgraph $P_{i,j}$ of P that is above v_j, and not above v_i. This is well defined since (P, V) satisfies the tree property. Let $P^1_{i,j}, \cdots P^k_{i,j}$ be k copies of $P_{i,j}$. Denote by v^t_i the vertices of $P^t_{i,j}$ corresponding to $v_i \in P_{i,j}$, and, similarly by v^t_j the vertices of $P^t_{i,j}$ corresponding to $v_j \in P_{i,j}$, for $t = 1, \cdots, k$.

2. identifying v_i^1 with v_i, v_j^t with v_i^{t+1} for $t = 1, \cdots, k-1$, and v_j^k with v_j.
3. updating the labeling of the vertices from the top to the bottom, accordingly to the firing of the transitions.

Note that $P_{i,j}^k$ admits a partition $V_{i,j}^k$ of its vertices, which is the image of V by the construction above. Note also that $(P_{i,j}^k, V_{i,j}^k)$ satisfies the tree property. For an illustration, see Figure 5.

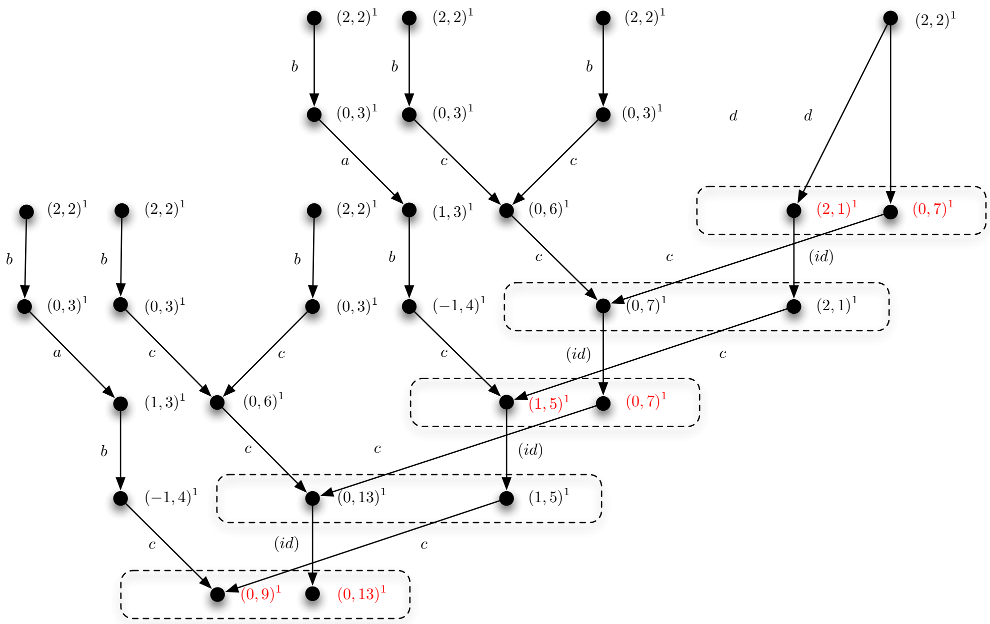

Figure 5: A 2-repetition on the promenade P_σ.

Now, Consider the promenade P_σ, and let $P(v_i), P(v_j) \in P(V_\sigma)$, such that

1. $P(v_i)$ is above $P(v_j)$ in P_σ, and
2. $v_i \in \mathcal{T}'_\sigma$ is labelled with the generalized configuration g_i, $v_j \in \mathcal{T}'_\sigma$ is labelled with the generalized configuration g_j, with $g_i \ll g_j$.

Since no ∞ multiplicity occurs in $\mathcal{T}'_\sigma$, g_j is necessarily obtained from g_i only by augmenting values for some coordinates. By construction, these augmentation coordinates are labelled with ∞ in g_j, and not in g_i (i.e. a lifting has been performed). Let a be the augmentation gap of this lifting. Assume $P(v_i)$ is labelled with the configuration c_i, and $P(v_j)$ by c_j. Then, remark the following:

- To any ∞ value for some coordinate in g_i corresponds a finite value t_i for the same coordinate in c_i. This ∞ value in g_i is inherited by construction on the same coordinate in g_j, and a finite value t_j corresponds to it in c_j. It may happen that $t_j < t_i$. It may even happen that $t_j < 0$. We denote t_i and t_j as finite coordinates of infinite heredity. See Figure 5 for an example.

- To any ∞ augmentation coordinate in g_j corresponds a finite value t_j for the same coordinate in c_j. Since it is an augmentation coordinate of the lifting, this coordinate has a finite value t_i in g_i and in c_i, and $t_i < t_j$. See also Figure 5.

Now, in order to raise the augmentation coordinates of this lifting to an arbitrary level $N \in \mathbb{N}$, it suffices to perform the k-repetition of the chain $P(v_i) \cdots P(v_j)$ in P_σ, for $k = \lceil \frac{N}{a} \rceil$. Of course, this k-repetition may decrease the values of the finite coordinates of infinite heredity, even to negative levels; it is therefore

necessary to ensure that these finite values are high enough when performing this k-repetition, i.e have been already leveled up by another k'-repetition above.
This yields the following inductive construction of a promenade of S, starting from P_σ:

The algorithm building the promenade:

Consider the promenade P returned by the following algorithm A:

input P_σ, $N \in \mathbb{N}$
$P \leftarrow P_\sigma$
$N_{local} \leftarrow N$
  For all $v(\sigma_j) \in \mathcal{T}'_\sigma$, from bottom to top, do
    For all $v(\sigma_i) \in \mathcal{T}'_\sigma$ above $v(\sigma_j)$, from bottom to top, do
    Let g_i be the label of σ_i
    Let g_j be the label of σ_j
      If $g_i \ll g_j$, let $a_{i,j}$ be the augmentation gap of the lifting
      $P \leftarrow$ the k-repetition of the chain $P(v(\sigma_i)) \cdots P(v(\sigma_j))$ in P, with $k = \lceil \frac{N_{local}}{a_{i,j}} \rceil$
      $N_{local} \leftarrow N_{local} + D_{i,j}$, where $D_{i,j}$ is the maximal decrease of finite coordinates of infinite heredity induced by the k-repetition of the step above
      end if
    end for
  end for
return P.

Then, clearly, P satisfies Theorem 3.6.

Introducing ∞ multiplicities

Augmentation multiplicities in $\mathcal{T}'_\sigma$ can be leveled up to an arbitrary level in a promenade P exactly the same way as augmentation coordinates, i.e. by performing as many k-repetition as needed. however, the situation for ∞ multiplicities in $\mathcal{T}'_\sigma$, and their respective counterparts in the promenade P_σ is more complicated than that of ∞ coordinates: as noted above, it may happen that coordinates of ∞ heredity in P_σ are decreased in a k-repetition, and reach negative values. The same applies for multiplicities of ∞ heredity, yet it is for now unclear how to deal with "negative multiplicities", for what that means. Another complication comes from the fact that, to a single generalized configuration in $\mathcal{T}'_\sigma$ of multiplicity greater than 1, several different single configurations (with different values for coordinates of ∞ heredity) can correspond in the promenade. One way to circumvent these difficulties is to merge all different configurations of the promenade corresponding to the same generalized configuration in $\mathcal{T}'_\sigma$ into only one, by arbitrarily decreasing coordinates of ∞ heredity to their minimal values among these different configurations. The one single configuration obtained can then be equipped with a "pseudo" multiplicity that we allow to take negative values. Such "pseudo" promenades with negative multiplicities will be used as intermediate steps in the inductive algorithm A, with the proviso that the resulting pseudo promenade has only positive multiplicities. Then, from this pseudo promenade, we will derive a promenade satisfying the theorem by undoing the merging of different configurations.
The definition of the "pseudo"-promenade Q_σ follows:

1. Let $z \in v(\sigma_k) \in \mathcal{T}'_\sigma$ of indegree 0, labelled with the single configuration c_z. Then, $v(\sigma_k) = \{z\}$. To z, we associate $Q(z) \in Q_\sigma$ labelled with c_z.
2. Let $z \in v(\sigma_k) \in \mathcal{T}'_\sigma$ of indegree greater than 1. Then, to z in $\mathcal{T}'_\sigma$, we associate $Q(z)$ in Q_σ, labelled with c_z, defined below.
   - If there exists $x \in v(\sigma_i) \in \mathcal{T}'_\sigma$, and an edge (x, z) in $\mathcal{T}'_\sigma$, labelled with $t \neq (join)$, there exists an edge $(Q(x), Q(z))$ labelled with t in Q_σ. Assume $Q(x)$ is labelled with c_x and let $c_{x,z} = (q, t_1^{x,z}, \cdots, t_m^{x,z})$ such that $c_x \rightarrow_t c_{x,z}$.

- If there exist $x \in v(\sigma_i) \in \mathcal{T'}_\sigma$, $y \in v(\sigma_j) \in \mathcal{T'}_\sigma$ and edges (x, z) and (y, z) in $\mathcal{T'}_\sigma$, labelled with $(join)$, there exist edges $(Q(x), Q(z))$ and $(Q(y), Q(z))$ labelled with $(join)$ in Q_σ. Assume $Q(x)$ is labelled with c_x, $Q(y)$ is labelled with c_y and let $c_{x,y,z} = (q, t_1^{x,y,z}, \cdots, t_m^{x,y,z})$ such that $\{c_x, c_y\} \rightarrow_t c_{x,y,z}$.

Then, $c_z = (q, t_1^z \cdots, t_m^z)$, where, for $k = 1, \cdots, m$, t_k^z is the minimal value of the $t_k^{x,z}$ and of the $t_k^{x,y,z}$ above.

`Repeating loops with augmentation configurations:`

Let Q be a pseudo-promenade of S, V be a partition of the vertices of Q such that (Q, V) satisfies the tree property. Let v_i above $v_j \in V$. Denote by c_i the configuration of S labeling v_i, and by c_j the configuration of S labeling v_j, and let $c'_j \subseteq c_j$ such that $c_i \ll c'_j$ and c'_j can be obtained from c_i by only modifying the values at some coordinates of some single configurations, with $v'_j \subseteq v_j$ the corresponding set of vertices. Then, the *k-repetition of the chain $v_i \cdots v_j$ in P*, for $k > 1$ is the promenade $P_{i,j}^k$ obtained from P by:

1. isolating the subgraph $P_{i,j}$ of P that is above v_j, and not above v_i. This is well defined since (P, V) satisfies the tree property. Let $P_{i,j}^1, \cdots P_{i,j}^k$ be k copies of $P_{i,j}$. Denote by v_i^t the vertices of $P_{i,j}^t$ corresponding to $v_i \in P_{i,j}$, by v_j^t the vertices of $P_{i,j}^t$ corresponding to $v_j \in P_{i,j}$, and by ${v'}_j^t$ the vertices of $P_{i,j}^t$ corresponding to $v'_j \in P_{i,j}$, for $t = 1, \cdots, k$.
2. identifying v_i^1 with v_i, ${v'}_j^t$ with v_i^{t+1} for $t = 1, \cdots, k-1$, and ${v'}_j^k$ with v'_j.
3. identifying also $v_j \setminus v'_j$ with $\cup_{t=1}^k v_j^t \setminus {v'}_j^t$.
4. updating the labeling of the vertices from the top to the bottom, accordingly to the firing of the transitions.

Now, the remark we have made on finite values of coordinates for the promenade P_σ above extend to multiplicities for the pseudo-promenade Q_σ as follows:
Let $Q(v_i), Q(v_j) \in Q(V_\sigma)$, such that

1. $Q(v_i)$ is above $Q(v_j)$ in Q_σ, and
2. $v_i \in \mathcal{T'}_\sigma$ is labelled with the generalized configuration g_i, $v_j \in \mathcal{T'}_\sigma$ is labelled with the generalized configuration g_j, with $g_i \ll g_j$.

By construction, g_j is obtained from g_i only by augmenting values for some coordinates, and adding generalized single configurations. By construction, these augmentation coordinates are labelled with ∞ in g_j, and not in g_i (i.e. a lifting has been performed), and the augmentation multiplicities are valued to ∞ in g_j and not in g_i (if the corresponding generalized single configuration appears in g_i, which may not be the case). Let a be the augmentation gap of this lifting. Assume $Q(v_i)$ is labelled with the configuration c_i, and $Q(v_j)$ by c_j.

- To any ∞ value for some coordinate in g_i corresponds a finite value t_i for the same coordinate in c_i. This ∞ value in g_i is inherited by construction on the same coordinate in g_j, and a finite value t_j corresponds to it in c_j. It may happen that $t_j < t_i$. It may even happen that $t_j < 0$. We denote t_i and t_j as finite coordinates of infinite heredity.

- To any ∞ value for some multiplicity in g_i corresponds a finite value k_i for the same multiplicity in c_i. This ∞ multiplicity in g_i is inherited by construction in g_j, and a finite multiplicity k_j corresponds to it in c_j. It may happen that $k_j < k_i$. It may even happen that $k_j < 0$. We denote k_i and k_j as finite multiplicities of infinite heredity.

- To any ∞ augmentation coordinate in g_j corresponds a finite value t_j for the same coordinate in c_j. Since it is an augmentation coordinate of the lifting, this coordinate has a finite value t_i in g_i and in c_i, and $t_i < t_j$.

- To any ∞ augmentation multiplicity in g_j corresponds a finite value k_j in c_j. If, in the derivation of $g_i \ll g_j$, this augmentation multiplicity has been introduced by adding new occurrences of a generalized configuration derived from g_i only by augmenting coordinates, to this multiplicity corresponds a finite multiplicity k_i in c_i, and $k_i < k_j$. In the other case (the augmentation multiplicity has been introduced by adding a new single generalized configuration in the derivation of $g_i \ll g_j$, no finite multiplicity corresponds to it in c_i.

Now, in order to raise the augmentation coordinates and augmentation multiplicities of this lifting to an arbitrary level $N \in \mathbb{N}$, it suffices to perform the k-repetition of the chain $Q(v_i) \cdots Q(v_j)$ in Q_σ, for $k = \lceil \frac{N}{a} \rceil$. Of course, this k-repetition may decrease the values of the finite coordinates and multiplicities of infinite heredity, even to negative levels; it is therefore necessary to ensure that these finite values are high enough when performing this k-repetition, i.e have been already leveled up by another k'-repetition above. This is ensured by the same algorithm A as above, by taking as input Q_σ and N.
Now, it is clear that the output of Algorithm A on Q_σ and N is a pseudo-promenade Q_σ^N with only positive coordinates and positive multiplicities. From Q_σ^N, we can inductively build a positive promenade P_σ^N, by starting from the top-most configurations of Q_σ^N, and firing from top to bottom the transitions labeling the edges of Q_σ^N. Then, clearly, the promenade P_σ^N obtained satisfies Theorem 3.6.

□

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